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Dilaton field induces a commutative Dp-brane coordinate*

B. Sazdović[a]

Institute of Physics, 11001 Belgrade, P.O. Box 57, Serbia

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Abstract. It is well known that the space-time coordinates x^μ and the corresponding Dp-brane world-volume become non-commutative when the ends of the open string are attached to a Dp-brane with the Neveu–Schwarz background field $B_{\mu\nu}$. In this paper, we extend these considerations by including an additional dilaton field Φ, linear in x^μ. In that case, the conformal part of the world-sheet metric becomes a new non-commutative variable, while the coordinate in the direction orthogonal to the hyper plane $\Phi = \text{const}$ becomes commutative.

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1 Introduction

Non-commutative theories arise as an effective description of the string theory in certain backgrounds [1]. In the presence of the antisymmetric tensor field $B_{\mu\nu}$, the quantization of the open string, whose ends are attached to a Dp-brane, leads to non-commutativity of the Dp-brane world-volume. For constant $B_{\mu\nu}$ and the space-time metric $G_{\mu\nu}$, this result can be obtained by using several different methods: the operator product expansion of the open string vertex operators [2], the mode expansion of the classical solution [3], the methods of the conformal field theory [4], and the Dirac quantization procedure for constrained systems [5].

In this paper, we keep the background fields $G_{\mu\nu}$ and $B_{\mu\nu}$ constant, but include an additional dilaton field Φ, linear in the space-time coordinate x^μ. This choice is consistent with the space-time field equations, obtained from the world-sheet conformal invariance. D-branes in the linear dilaton background were studied in [6,7]: in [6], the Dirichlet boundary condition was constructed, while in [7], the non-commutativity structure was analyzed.

In our approach, the conformal part F of the world-sheet metric is a dynamical variable. Hence, beside the known boundary conditions corresponding to the Dp-brane coordinates x^i, we have an additional condition corresponding to F. The authors of [7] fixed F, and consequently, they missed the additional boundary condition, whereby their treatment lost generality, as we will see later.

In this paper, we apply the canonical method along the lines of [5], and treat our boundary conditions as the canonical constraints. They appear as particular orbifold conditions, so that all the effective variables are symmetric under the transformation $\sigma \to -\sigma$, which reduces the number of phase space variables by a factor of two.

All the constraints are of second class. Instead of using the Dirac brackets, as in [5], we shall explicitly solve the constraints in terms of the effective open string variables: the coordinates q^i and the conformal part of the world-sheet metric f.

Expressed in terms of the open string variables, the effective theory is found to have exactly the same form as the original theory, but with different Dp-brane background fields. Moreover, $B_{\mu\nu}$ contributes only to the effective metric tensor, as in the absence of Φ, and the effective dilaton field is linear in q^i.

After calculating the Poisson brackets of all the dynamical variables, one finds that, effectively, $B_{\mu\nu}$ is projected onto the directions orthogonal to $\partial_\mu \Phi$. Consequently, on the world-sheet boundary, there exists a Dp-brane coordinate, defined by $x \equiv x^\mu \partial_\mu \Phi$, which commutes with all the other coordinates. However, it turns out that F does not commute with the string coordinates orthogonal to $\partial_\mu \Phi$. Therefore, when the linear dilaton is present in the theory, and if F is interpreted as an additional coordinate, the total number of non-commuting coordinates remains the same.

2 Definition of the model and canonical analysis

We study the theory defined by the action

$$S = \kappa \int_\Sigma \mathrm{d}^2\xi \sqrt{-g}$$

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[a] e-mail: sazdovic@phy.bg.ac.yu

$$\times \left\{ \left[\frac{1}{2} g^{\alpha\beta} G_{\mu\nu}(x) + \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu}(x) \right] \partial_\alpha x^\mu \partial_\beta x^\nu + \Phi(x) R^{(2)} \right\} + 2\kappa \int_{\partial\Sigma} A_i \mathrm{d}x^i, \tag{2.1}$$

which describes the propagation of the bosonic open string [8–12]. Here, ξ^α $(\alpha = 0, 1)$ are the coordinates of the two-dimensional world-sheet Σ, $x^\mu(\xi)$ $(\mu = 0, 1, ..., D-1)$ are the coordinates of the D-dimensional space-time M_D, x^i $(i = 0, 1, ..., p)$ are the coordinates describing the Dp-brane in our gauge choice, $g_{\alpha\beta}$ is the intrinsic world-sheet metric, $R^{(2)}$ is the corresponding scalar curvature, and we use the notation $\partial_\alpha \equiv \partial/\partial\xi^\alpha$, $\partial_\mu \equiv \partial/\partial x^\mu$ and $\partial_i \equiv \partial/\partial x^i$. The non-trivial background is defined by the space-time metric tensor $G_{\mu\nu}$, the antisymmetric tensor $B_{\mu\nu} = -B_{\nu\mu}$, the dilaton Φ, and the $U(1)$ gauge field A_i, living on the Dp-brane.

If both ends of the open string are attached to the same Dp-brane, the action can be written as

$$S = \kappa \int_\Sigma \mathrm{d}^2\xi \sqrt{-g} \times \left\{ \left[\frac{1}{2} g^{\alpha\beta} G_{\mu\nu}(x) + \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}} \mathcal{F}_{\mu\nu}(x) \right] \partial_\alpha x^\mu \partial_\beta x^\nu + \Phi(x) R^{(2)} \right\}, \tag{2.2}$$

where $\mathcal{F}_{\mu\nu}$ is the modified Born–Infeld field strength,

$$\mathcal{F}_{\mu\nu} = B_{\mu\nu} + (\partial_i A_j - \partial_j A_i)\delta^i_\mu \delta^j_\nu. \tag{2.3}$$

In the conformal gauge $g_{\alpha\beta} = \mathrm{e}^{2F}\eta_{\alpha\beta}$, we have $R^{(2)} = 2\Delta F$, and the action takes the form

$$S = \kappa \int_\Sigma \mathrm{d}^2\xi \left\{ \left[\frac{1}{2}\eta^{\alpha\beta} G_{\mu\nu}(x) + \varepsilon^{\alpha\beta}\mathcal{F}_{\mu\nu}(x) \right] \partial_\alpha x^\mu \partial_\beta x^\nu + 2\Phi(x)\mathrm{e}^{2F}\Delta F \right\}. \tag{2.4}$$

Note that the dilaton field breaks the conformal invariance; hence, the conformal part of the metric survives, and the dynamical variables of the theory are x^μ and F.

It is an enormous task to make further progress with arbitrary background fields. Instead, we employ a particular solution of the space-time field equations [9]:

$$\beta^G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{4}\mathcal{F}_{\mu\rho\sigma}\mathcal{F}_\nu{}^{\rho\sigma} + 2D_\mu a_\nu = 0, \tag{2.5}$$

$$\beta^{\mathcal{F}}_{\mu\nu} \equiv D_\rho \mathcal{F}^\rho_{\mu\nu} - 2a_\rho \mathcal{F}^\rho_{\mu\nu} = 0, \tag{2.6}$$

$$\beta^\Phi \equiv 4\pi\kappa\frac{D-26}{3} - R + \tfrac{1}{12}\mathcal{F}_{\mu\rho\sigma}\mathcal{F}^{\mu\rho\sigma} - 4D_\mu a^\mu + 4a^2 = 0. \tag{2.7}$$

These equations are obtained as a consequence of the quantum world-sheet conformal invariance, which is a necessary condition for the consistency of the theory. Here, $a_\mu = \partial_\mu\Phi$, $\mathcal{F}_{\mu\rho\sigma}$ is the field strength for $\mathcal{F}_{\mu\nu}$, and $R_{\mu\nu}$, R and D_μ are Ricci tensor, scalar curvature and the covariant derivative corresponding to the Riemannian structure of space-time. Following [12], we chose

$$G_{\mu\nu}(x) = G_{\mu\nu} = \text{const}, \quad \mathcal{F}_{\mu\nu}(x) = \mathcal{F}_{\mu\nu} = \text{const},$$

$$\Phi(x) = \Phi_0 + a_\mu x^\mu \quad (a_\mu = \text{const}), \tag{2.8}$$

which is an exact solution if

$$a^2 = \kappa\pi\frac{26-D}{3}. \tag{2.9}$$

We assume, for simplicity, that $B_{\mu\nu}$ and a_μ are non-trivial only along the Dp-brane directions, so that $\mathcal{F}_{\mu\nu} \to \mathcal{F}_{ij}$ and $a_\mu \to a_i$. We also chose the coordinates so that $G_{\mu\nu} = 0$ for $\mu = i \in \{0, 1, ..., p\}$ and $\nu = a \in \{p+1, ..., D-1\}$. In that case, the action integral takes the form

$$S = \kappa \int_\Sigma \mathrm{d}^2\xi \left\{ \frac{1}{2}\eta^{\alpha\beta} G_{\mu\nu}\partial_\alpha x^\mu \partial_\beta x^\nu + \varepsilon^{\alpha\beta}\mathcal{F}_{ij}\partial_\alpha x^i \partial_\beta x^j + 2(\Phi_0 + a_i x^i)\mathrm{e}^{2F}\Delta F \right\}, \tag{2.10}$$

and the components x^a decouple from all the other variables.

In order to apply the canonical method to the action (2.10), we briefly review the results of the canonical analysis given in [13], adapted to the present case. The currents corresponding to the Dp-brane directions have the form

$$J^i_\pm = P^{Tij} j_{\pm j} + \frac{a^i}{2a^2} i^\Phi_\pm = j^i_\pm - \frac{a^i}{a^2} j \quad (a_i \equiv \partial_i \Phi), \tag{2.11}$$

$$j = a^i j_{\pm i} - \frac{1}{2} i^\Phi_\pm, \quad i^\Phi_\pm = \pi \pm 2\kappa a_i x^{i\prime}, \tag{2.12}$$

$$j_{\pm i} = \pi_i + 2\kappa\Pi_{\pm ij} x^{j\prime} \quad \left(\Pi_{\pm ij} \equiv \mathcal{F}_{ij} \pm \frac{1}{2} G_{ij} \right), \tag{2.13}$$

where π_i and π are the canonical momenta corresponding to x^i and F, respectively. For the directions orthogonal to the Dp-brane, the only non-trivial current is

$$j_{\pm a} = \pi_a \pm \kappa G_{ab} x^{b\prime}, \tag{2.14}$$

where π_a is the momentum corresponding to x^a. It commutes with all the other currents, and will be ignored in what follows. We also introduce the projection operators

$$P^{\mathrm{L}}_{ij} = \frac{a_i a_j}{a^2}, \quad P^{\mathrm{T}}_{ij} = G_{ij} - \frac{a_i a_j}{a^2}. \tag{2.15}$$

All τ and σ derivatives of x^i and F can be expressed in terms of the corresponding currents:

$$\dot{x}^i = \frac{1}{2\kappa}(J^i_- + J^i_+), \quad \dot{F} = \frac{1}{4\kappa}(i^F_- + i^F_+), \tag{2.16}$$

$$x^{i\prime} = \frac{1}{2\kappa}(J^i_+ - J^i_-), \quad F' = \frac{1}{4\kappa}(i^F_+ - i^F_-). \tag{2.17}$$

The Dp-brane part of the canonical Hamiltonian density

$$\mathcal{H}_\mathrm{c} = T_- - T_+ \tag{2.18}$$

is defined in terms of the energy-momentum tensor components

$$T_\pm = \mp\frac{1}{4\kappa}\left(G^{ij} J_{\pm i} J_{\pm j} + \frac{j}{a^2} i^\Phi_\pm \right) + \frac{1}{2}(i^{\Phi\prime}_\pm - F' i^\Phi_\pm). \tag{2.19}$$

These components satisfy two independent copies of the Virasoro algebra:

$$\{T_\pm, T_\pm\} = -[T_\pm(\sigma) + T_\pm(\bar\sigma)]\delta', \quad \{T_\pm, T_\mp\} = 0. \tag{2.20}$$

3 Open string boundary conditions as constraints

To describe the open string temporal evolution, we need both the equations of motion and the boundary conditions. The canonical world-sheet field equations were derived in [13]. In the particular case of the present paper, they have the form $\Delta x^\mu - 2a^\mu \Delta F = 0$, $a_\mu \Delta x^\mu = 0$, and for $a^2 \neq 0$ they take the standard form:

$$\Delta x^\mu = 0, \quad \Delta F = 0. \tag{3.1}$$

In order to study the boundary conditions, it is useful to introduce the variables

$$\gamma_i^{(0)} \equiv \frac{\delta S}{\delta x'^i} = \kappa(-G_{ij}x^{j\prime} + 2\mathcal{F}_{ij}\dot{x}^j - 2a_i F'),$$
$$\gamma^{(0)} \equiv \frac{\delta S}{\delta F'} = -2\kappa a_i x^{i\prime}. \tag{3.2}$$

For the variables x^i and F, we use Neumann boundary conditions, allowing arbitrary variations δx^i and δF on the string end points. Then, the boundary conditions can be written in the form

$$\gamma_i^{(0)}\Big|_{\partial\Sigma} = 0, \quad \gamma^{(0)}\Big|_{\partial\Sigma} = 0. \tag{3.3}$$

The second condition, corresponding to the additional variable F, is a new one, as compared to the dilaton free case. Note that the constant field $\mathcal{F}_{ij}$ does not appear in the equations of motion, and contributes only to the boundary conditions.

For the variables x^a, we use Dirichlet boundary conditions, requiring the edges of the string to be fixed, $\delta x^a|_{\partial\Sigma} = 0$.

Using the expressions for τ and σ derivatives, (2.16) and (2.17), we can rewrite the boundary conditions (3.3) in terms of the currents,

$$\gamma_i^{(0)} = \Pi_{+ij}J_-^j + \Pi_{-ij}J_+^j + \frac{a_i}{2}(i_-^F - i_+^F),$$
$$\gamma^{(0)} = \frac{1}{2}(i_-^\Phi - i_+^\Phi). \tag{3.4}$$

Following the approach of [5], we consider the expressions $\gamma_i^{(0)}\Big|_{\partial\Sigma}$ and $\gamma^{(0)}\big|_{\partial\Sigma}$ as the canonical constraints. In order to find the corresponding consistency conditions, we note that the background fields $G_{ij}, \mathcal{F}_{ij}$ and a_i are x^i-independent, which simplifies the form of the Poisson brackets:

$$\{H_c, J_{\pm i}\} = \mp J'_{\pm i}, \quad \{H_c, i_\pm^\Phi\} = \mp i_\pm'^\Phi,$$
$$\{H_c, i_\pm^F\} = \mp i_\pm'^F. \tag{3.5}$$

The Dirac consistency procedure generates two infinite sets of conditions, $\gamma_i^{(n)}\Big|_{\partial\Sigma} = 0$ and $\gamma^{(n)}\big|_{\partial\Sigma} = 0$, $(n \geq 1)$, where

$$\gamma_i^{(n)} \equiv \{H_c, \gamma_i^{(n-1)}\}$$
$$= \partial_\sigma^n \Big\{\Pi_{+ij}J_-^j + (-1)^n \Pi_{-ij}J_+^j + \frac{a_i}{2}\left[i_-^F - (-1)^n i_+^F\right]\Big\}, \tag{3.6}$$

$$\gamma^{(n)} \equiv \{H_c, \gamma^{(n-1)}\} = \frac{1}{2}\partial_\sigma^n\left[i_-^\Phi - (-1)^n i_+^\Phi\right]. \tag{3.7}$$

Using the Taylor expansion formula, we find that these conditions at $\sigma = 0$ can be compactly written in the form

$$\Gamma_i(\sigma) \equiv \sum_{n\geq 0}\frac{\sigma^n}{n!}\gamma_i^{(n)}(0)$$
$$= \Pi_{+ij}J_-^j(\sigma) + \Pi_{-ij}J_+^j(-\sigma) + \frac{a_i}{2}\left[i_-^F(\sigma) - i_+^F(-\sigma)\right], \tag{3.8}$$

$$\Gamma(\sigma) \equiv \sum_{n\geq 0}\frac{\sigma^n}{n!}\gamma^{(n)}(0)$$
$$= \frac{1}{2}\left[i_-^\Phi(\sigma) - i_+^\Phi(-\sigma)\right]. \tag{3.9}$$

Similarly, the conditions at $\sigma = \pi$ are given as

$$\bar{\Gamma}_i(\sigma) \equiv \sum_{n\geq 0}\frac{(\sigma-\pi)^n}{n!}\gamma_i^{(n)}(\pi)$$
$$= \Pi_{+ij}J_-^j(\sigma) + \Pi_{-ij}J_+^j(2\pi - \sigma) + \frac{a_i}{2}\left[i_-^F(\sigma) - i_+^F(2\pi-\sigma)\right], \tag{3.10}$$

$$\bar{\Gamma}(\sigma) \equiv \sum_{n\geq 0}\frac{(\sigma-\pi)^n}{n!}\gamma^{(n)}(\pi)$$
$$= \frac{1}{2}\left[i_-^\Phi(\sigma) - i_+^\Phi(2\pi-\sigma)\right]. \tag{3.11}$$

These expressions differ from the boundary conditions (3.4) only in the arguments of the positive chirality currents: σ is replaced by $-\sigma$ in the first case, and by $2\pi - \sigma$ in the second case. From (3.8)–(3.11), we conclude that all positive chirality currents, and consequently all the variables, are periodic in σ, with period 2π.

Equations (3.5) imply that all constraints weakly commute with the Hamiltonian:

$$\{H_c, \Gamma_i(\sigma)\} = \Gamma_i'(\sigma), \quad \{H_c, \Gamma(\sigma)\} = \Gamma'(\sigma). \tag{3.12}$$

Therefore, there are no more constraints, and the consistency procedure is completed. A straightforward calculation yields

$$\{\Gamma_i(\sigma), \Gamma_j(\bar\sigma)\} = -\kappa\tilde{G}_{ij}\delta'(\sigma - \bar\sigma),$$
$$\{\Gamma(\sigma), \Gamma(\bar\sigma)\} = 0, \tag{3.13}$$
$$\{\Gamma_i(\sigma), \Gamma(\bar\sigma)\} = -2\kappa a_i\delta'(\sigma - \bar\sigma), \tag{3.14}$$

where we introduced the effective metric tensor

$$\tilde{G}_{ij} \equiv G_{ij} - 4\mathcal{F}_{ik}P^{Tkq}\mathcal{F}_{qj}. \tag{3.15}$$

Following [4], we refer to $\tilde{G}_{ij}$ as the open string metric tensor – the metric tensor seen by the open string. When the inverse effective metric $\tilde{G}^{ij}$ is used to raise the index of V_i, we write $\tilde{V}^i = \tilde{G}^{ij} V_j$, and similarly, $\tilde{V}^2 = \tilde{G}^{ij} V_i V_j$. We also preserve the standard notation, $V^i = G^{ij} V_j$ and $V^2 = G^{ij} V_i V_j$.

Introducing the compact notation $\Gamma_A = \{\Gamma_i, \Gamma\}$ for the complete set of constraints, we find

$$\{\Gamma_A(\sigma), \Gamma_B(\bar\sigma)\} = -\kappa \begin{vmatrix} \tilde{G}_{ij} & 2a_i \\ 2a_j & 0 \end{vmatrix} \delta'(\sigma - \bar\sigma) \equiv \Delta_{AB} \delta'(\sigma - \bar\sigma), \tag{3.16}$$

and

$$\triangle \equiv \det \triangle_{AB} = -4(-\kappa)^{p+2} \tilde{a}^2 \det \tilde{G}_{ij}. \tag{3.17}$$

We assume that $\tilde{a}^2 \neq 0$. In that case, $\operatorname{rank} \triangle_{AB} = p + 2$ and all the constraints are of the second class (except for the zero mode [14]).

4 Solving the boundary conditions

The periodicity condition solves the second set of constraints (3.10)–(3.11). In order to solve the first set (3.8)–(3.9), it is useful to introduce the new variables

$$q^i(\sigma) = \frac{1}{2}\left[x^i(\sigma) + x^i(-\sigma)\right], \quad \bar{q}^i(\sigma) = \frac{1}{2}\left[x^i(\sigma) - x^i(-\sigma)\right], \tag{4.1}$$

$$p_i(\sigma) = \frac{1}{2}\left[\pi_i(\sigma) + \pi_i(-\sigma)\right], \quad \bar{p}_i(\sigma) = \frac{1}{2}\left[\pi_i(\sigma) - \pi_i(-\sigma)\right], \tag{4.2}$$

$$f(\sigma) = \frac{1}{2}\left[F(\sigma) + F(-\sigma)\right], \quad \bar{f}(\sigma) = \frac{1}{2}\left[F(\sigma) - F(-\sigma)\right], \tag{4.3}$$

$$p(\sigma) = \frac{1}{2}\left[\pi(\sigma) + \pi(-\sigma)\right], \quad \bar{p}(\sigma) = \frac{1}{2}\left[\pi(\sigma) - \pi(-\sigma)\right], \tag{4.4}$$

which we call the open string variables. Using the relations

$$\frac{1}{2}[j_{-i}(\sigma) + j_{+i}(-\sigma)] = p_i + 2\kappa \mathcal{F}_{ij} \bar{q}^{j\prime} - \kappa G_{ij} q^{j\prime}, \tag{4.5}$$

$$\frac{1}{2}[j_{-i}(\sigma) - j_{+i}(-\sigma)] = \bar{p}_i + 2\kappa \mathcal{F}_{ij} q^{j\prime} - \kappa G_{ij} \bar{q}^{j\prime}, \tag{4.6}$$

$$\frac{1}{2}[i^{\Phi}_{-}(\sigma) - i^{\Phi}_{+}(-\sigma)] = \bar{p} - 2\kappa a_i \bar{q}^{i\prime}, \tag{4.7}$$

we can write the constraints in terms of the open string variables

$$\Gamma_i(\sigma) \tag{4.8}$$

$$= 2(\mathcal{F}P^{\mathrm{T}})_i{}^j p_j + \bar{p}_i + \frac{1}{a^2} \mathcal{F}_{ij} a^j p - \kappa \tilde{G}_{ij} \bar{q}^{j\prime} - 2\kappa a_i \bar{f}',$$

$$\Gamma(\sigma) = \bar{p} - 2\kappa a_i \bar{q}^{i\prime}. \tag{4.9}$$

The parts of these relations which are symmetric and antisymmetric under $\sigma \to -\sigma$ separately vanish. Therefore, the conditions $\Gamma_i(\sigma) = 0$ and $\Gamma(\sigma) = 0$ imply

$$\bar{p}_i = 0, \quad 2(\mathcal{F}P^{\mathrm{T}})_i{}^j p_j + \frac{1}{a^2} \mathcal{F}_{ij} a^j p - \kappa \tilde{G}_{ij} \bar{q}^{j\prime} - 2\kappa a_i \bar{f}' = 0, \tag{4.10}$$

$$\bar{p} = 0, \quad a_i \bar{q}^{i\prime} = 0. \tag{4.11}$$

Now, we can solve the antisymmetric (barred) variables in terms of the symmetric ones,

$$\bar{p}_i = 0, \quad \bar{q}^{i\prime} = -2(\Theta^{ij} p_j + \Theta^i p), \tag{4.12}$$

$$\bar{p} = 0, \quad \bar{f}' = 2\Theta^i p_i, \tag{4.13}$$

where

$$\Theta^{ij} = \frac{-1}{\kappa} \tilde{P}^{\mathrm{T}ik} \mathcal{F}_{kq} P^{\mathrm{T}qj} \quad (\Theta^{ij} = -\Theta^{ji}), \tag{4.14}$$

$$\Theta^i = \frac{(\tilde{a}\mathcal{F})^i}{2\kappa \tilde{a}^2} = \frac{(a\mathcal{F}\tilde{G}^{-1})^i}{2\kappa a^2}. \tag{4.15}$$

In analogy with (2.15), we introduced the tilde projectors:

$$\tilde{P}^{\mathrm{L}ij} = \frac{\tilde{a}^i \tilde{a}^j}{\tilde{a}^2}, \quad \tilde{P}^{\mathrm{T}ij} = \tilde{G}^{ij} - \frac{\tilde{a}^i \tilde{a}^j}{\tilde{a}^2}. \tag{4.16}$$

Using (4.1)–(4.4) and (4.12)–(4.13), we can express the original variables in terms of the new ones,

$$x^i = q^i - 2\int^{\sigma} \mathrm{d}\sigma_1 \left(\Theta^{ij} p_j + \Theta^i p\right), \quad \pi_i = p_i, \tag{4.17}$$

$$F = f + 2\,\Theta^i \int^{\sigma} \mathrm{d}\sigma_1\, p_i, \quad \pi = p. \tag{4.18}$$

As a consequence of the particular form of the conditions $\Gamma_i(\sigma) = 0$ and $\Gamma(\sigma) = 0$, the effective theory depends only on the variables symmetric under $\sigma \to -\sigma$.

5 The effective theory in terms of the open string variables

The original string theory is completely described by the energy-momentum tensor $T_{\pm}$, (2.19), in terms of the variables x^i, F and the corresponding momenta π_i, π. We now wish to find the effective energy-momentum tensor $\tilde{T}_{\pm}$ in terms of the new variables q^i, f and their momenta p_i, p.

Since the form of the energy-momentum tensor depends on the currents, let us first express the currents in terms of the new variables. In analogy with (2.11)–(2.13), we introduce the set of new, open string currents:

$$\tilde{i}^{\Phi}_{\pm} = p \pm 2\kappa a_i q^{i\prime}, \tag{5.1}$$

$$\tilde{j}_{\pm i} = p_i \pm \kappa \tilde{G}_{ij} q^{j\prime}, \quad \tilde{j} = \tilde{a}^i \tilde{j}_{\pm i} - \frac{1}{2} \tilde{i}^{\Phi}_{\pm}, \tag{5.2}$$

$$\tilde{J}^i_{\pm} = \tilde{P}^{\mathrm{T}ij} \tilde{j}_{\pm j} + \frac{\tilde{a}^i}{2\tilde{a}^2} \tilde{i}^{\Phi}_{\pm} = \tilde{G}^{ij} \tilde{j}_{\pm i} - \frac{\tilde{a}^i}{\tilde{a}^2} \tilde{j}. \tag{5.3}$$

They depend on the new variables similarly as the original currents depend on the original variables. The metric tensor is systematically replaced by the effective one. The main difference is that there is no explicit dependence on $B_{\mu\nu}$; it contributes only through the effective metric tensor. Formally, we can first put $\mathcal{F}_{ij} \to 0$ and then $G_{ij} \to \tilde{G}_{ij}$.

With the help of (4.17)–(4.18), we can express the original currents (2.11)–(2.13) in terms of the open string currents (5.1)–(5.3):

$$i^{\Phi}_{\pm} = \tilde{i}^{\Phi}_{\pm}, \quad \frac{j}{a^2} = \frac{\tilde{j}}{\tilde{a}^2} + \frac{2\kappa}{a^2} a^i \mathcal{F}_{ij} q^{j\prime}, \tag{5.4}$$

$$J_{\pm i} = \pm 2 \tilde{\Pi}_{\pm ij} \tilde{J}^j_{\pm} \quad \left(\tilde{\Pi}_{\pm ij} = \Pi_{\pm ij} - P^{\mathrm{L}}{}_i{}^k \mathcal{F}_{kj} \right). \tag{5.5}$$

Using the identity

$$4 G^{ij} \tilde{\Pi}_{\pm ik} \tilde{\Pi}_{\pm jq} = \tilde{G}_{kq} \pm 2 (\mathcal{F}_{kr} P^{\mathrm{L}r}{}_q - P^{\mathrm{L}}{}_k{}^r \mathcal{F}_{rq}), \tag{5.6}$$

we obtain a useful relation,

$$G^{ij} J_{\pm i} J_{\pm j} = \tilde{G}_{ij} \tilde{J}^i_{\pm} \tilde{J}^j_{\pm} \mp \frac{2}{\tilde{a}^2} \tilde{i}^{\Phi}_{\pm} \tilde{a}^i \mathcal{F}_{ij} \tilde{j}^j_{\pm}. \tag{5.7}$$

Finally, we are in a position to find the energy-momentum tensor in terms of the open string variables. With the help of (5.4)–(5.7) we obtain

$$T_{\pm} = \tilde{T}_{\pm}, \tag{5.8}$$

where

$$\tilde{T}_{\pm} = \mp \frac{1}{4\kappa} \left(\tilde{G}^{ij} \tilde{J}_{\pm i} \tilde{J}_{\pm j} + \frac{\tilde{j}}{\tilde{a}^2} \tilde{i}^{\Phi}_{\pm} \right) + \frac{1}{2} (\tilde{i}^{\Phi\prime}_{\pm} - f' \tilde{i}^{\Phi}_{\pm}). \tag{5.9}$$

Thus, we can conclude that the effective energy-momentum tensor depends on the open string currents in exactly the same way as the original energy-momentum tensor depends on the original currents.

In the standard formulation, the theory is expressed in terms of the canonical variables x^i, F, π_i and π, in the background described by the fields G_{ij}, $\mathcal{F}_{ij}$ and Φ. In that case, together with the equations of motion, the boundary conditions (3.3) must be used.

The effective theory is expressed in terms of the new canonical variables q^i, f, p_i and p, in the background described by the fields $\tilde{G}_{ij}, \tilde{\mathcal{F}}_{ij} = 0$ and $\tilde{\Phi} = \Phi_0 + a_i q^i$. In this case, the symmetries under the transformations $\sigma \to \sigma + 2\pi$ and $\sigma \to -\sigma$ should be imposed, which are particular forms of the orbifold conditions.

The open string Hamiltonian and the corresponding equations of motion take the form

$$\tilde{\mathcal{H}}_{\mathrm{c}} = \tilde{T}_- - \tilde{T}_+, \qquad \tilde{\Delta} q^i = 0, \quad \tilde{\Delta} f = 0. \tag{5.10}$$

The Laplace operator $\tilde{\Delta}$ is defined with respect to the effective world-sheet metric $\tilde{g}_{\alpha\beta} = \mathrm{e}^{2f} \eta_{\alpha\beta}$.

6 The non-commutative conformal factor and a commutative Dp-brane direction

From the standard Poisson brackets

$$\{x^i(\sigma), \pi_j(\bar{\sigma})\} = \delta^i_j \delta(\sigma - \bar{\sigma}),$$
$$\{F(\sigma), \pi(\bar{\sigma})\} = \delta(\sigma - \bar{\sigma}) \tag{6.1}$$

and the relations (4.1)–(4.4), we have

$$\{q^i(\sigma), p_j(\bar{\sigma})\} = \delta^i_j \delta_{\mathrm{s}}(\sigma, \bar{\sigma}),$$
$$\{f(\sigma), p(\bar{\sigma})\} = \delta_{\mathrm{s}}(\sigma, \bar{\sigma}), \tag{6.2}$$

where

$$\delta_{\mathrm{s}}(\sigma, \bar{\sigma}) = \frac{1}{2} \left[\delta(\sigma - \bar{\sigma}) + \delta(\sigma + \bar{\sigma}) \right] \quad (\sigma, \bar{\sigma} \in [0, \pi]) \tag{6.3}$$

is the symmetric delta-function. Consequently, on the subspace symmetric under $\sigma \to -\sigma$, p_i and p are canonically conjugate momenta to the variables q^i and f, respectively.

With the help of (4.17)–(4.18), we calculate the Poisson brackets

$$\{x^i(\sigma), x^j(\bar{\sigma})\} = 2\Theta^{ij} \Delta(\sigma + \bar{\sigma}),$$
$$\{x^i(\sigma), F(\bar{\sigma})\} = 2\Theta^i \Delta(\sigma + \bar{\sigma}), \tag{6.4}$$

where Θ^{ij} and Θ^i have been defined in (4.14) and (4.15), respectively, and

$$\Delta(\sigma + \bar{\sigma}) = \theta(\sigma + \bar{\sigma}) = \begin{cases} 0, \ \sigma = 0 = \bar{\sigma}, \\ 1, \ \sigma = \pi = \bar{\sigma}, \\ \frac{1}{2}, \ \text{otherwise.} \end{cases} \tag{6.5}$$

If we separate the center of mass variable by introducing $x^i(\sigma) = x^i_{\mathrm{cm}} + X^i(\sigma)$, where $x^i_{\mathrm{cm}} = \frac{1}{\pi} \int_0^{\pi} \mathrm{d}\sigma x^i(\sigma)$, we have

$$\{X^i(\sigma), X^j(\bar{\sigma})\} \tag{6.6}$$
$$= 2\Theta^{ij} \left[\Delta(\sigma + \bar{\sigma}) - \frac{1}{2} \right] = \Theta^{ij} \begin{cases} -1, \ \sigma = 0 = \bar{\sigma} \\ 1, \ \sigma = \pi = \bar{\sigma} \\ 0, \ \text{otherwise} \end{cases},$$

$$\{X^i(\sigma), F(\bar{\sigma})\} \tag{6.7}$$
$$= 2\Theta^i \left[\Delta(\sigma + \bar{\sigma}) - \frac{1}{2} \right] = \Theta^i \begin{cases} -1, \ \sigma = 0 = \bar{\sigma}, \\ 1, \ \sigma = \pi = \bar{\sigma}, \\ 0, \ \text{otherwise.} \end{cases}$$

The relation (6.7) has not been considered before in the literature. It shows that, in the presence of a dilaton, the non-commutativity between X^i and F appears on the world-sheet boundary. The expression for this new non-commutativity parameter, Θ^i, is proportional to the Born–Infeld field, $\mathcal{F}_{ij}$.

The relation (6.6) has the same form as in the absence of a dilaton [3–7], but there are some significant differences.

Let us first explain the geometrical meaning of the projectors $P^{\mathrm{T}ij}$ and $\tilde{P}^{\mathrm{T}ij}$. Note that the vector a_i is normal to the p-dimensional submanifold M_p of the Dp-brane, defined by the condition $\Phi(x) = \text{const}$. For $a^2 \neq 0$ ($\tilde{a}^2 \neq 0$), the corresponding unit vectors for the closed and open string are $n_i = a_i/\sqrt{\varepsilon a^2}$ and $\tilde{n}_i = a_i/\sqrt{\tilde{\varepsilon}\tilde{a}^2}$, respectively. Here $\varepsilon = 1$ ($\tilde{\varepsilon} = 1$) if a_i is timelike, and $\varepsilon = -1$ ($\tilde{\varepsilon} = -1$) if a_i is spacelike with respect to the metric G_{ij} ($\tilde{G}_{ij}$). Consequently, the induced metrics on M_p are

$$P^{\mathrm{T}}{}_{ij} = G_{ij} - \varepsilon n_i n_j \equiv G^{(p)}_{ij}, \qquad \tilde{P}^{\mathrm{T}}{}_{ij} = \tilde{G}_{ij} - \tilde{\varepsilon}\tilde{n}_i\tilde{n}_j \equiv \tilde{G}^{(p)}_{ij}, \tag{6.8}$$

and we can rewrite (4.14) in the form

$$\Theta^{ij} = \frac{-1}{\kappa}\tilde{G}^{(p)ik}\mathcal{F}_{kq}G^{(p)qj}. \tag{6.9}$$

This expression has similar form as in the absence of dilaton. Again, the essential part is the Born–Infeld field strength $\mathcal{F}_{kq}$, but in the present case, we raise indices with the induced metrics $\tilde{G}^{(p)ij}$ and $G^{(p)ij}$ on M_p, instead of the metrics $G^{ij}_{\text{eff}} = (G - 4\mathcal{F}G^{-1}\mathcal{F})^{-1ij}$ and G^{ij} on the Dp-brane.

From the relations $a_i P^{\mathrm{T}ij} = 0$ and $\tilde{a}\mathcal{F}a = 0$, it follows that $a_i\Theta^{ij} = 0$ and $a_i\Theta^i = 0$, so that the component $x \equiv a_i x^i$ commutes with all the other coordinates as well as with F:

$$\{x(\sigma), x^j(\bar{\sigma})\} = 0, \quad \{x(\sigma), F(\bar{\sigma})\} = 0. \tag{6.10}$$

This is an example of the Dp-brane with one commutative coordinate in the a_i direction.

7 Concluding remarks

In the present paper, we studied the string propagation in the presence of the linear dilaton field, in addition to the constant $G_{\mu\nu}$ and $B_{\mu\nu}$. This choice of the background preserves the conformal symmetry at the quantum level. We investigated the contribution of the dilaton field to the non-commutativity of the Dp-brane world-volume.

The initial open string boundary conditions produce an infinite set of constraints, obtained by applying the Dirac consistency procedure. We solved them explicitly by imposing the periodicity condition and expressing the odd variables (with respect to $\sigma \to -\sigma$) in terms of the even ones.

The effective theory, given in terms of the open string variables q^j and f, has precisely the same form as the original theory expressed in terms of the closed string variables x^j and F, including the energy-momentum tensor, the Hamiltonian and the field equations. There are only two differences. First, the closed string background G_{ij}, $\mathcal{F}_{ij} = B_{ij} + \partial_i A_j - \partial_j A_i$ and $\Phi = \Phi_0 + a_i x^i$ is replaced by the open string one:

$$G_{ij} \to \tilde{G}_{ij} = G_{ij} - 4\mathcal{F}_{ik}P^{\mathrm{T}kq}\mathcal{F}_{qj}, \quad \mathcal{F}_{ij} \to \tilde{\mathcal{F}}_{ij} = 0,$$

$$\Phi \to \tilde{\Phi} = \Phi_0 + a_i q^i. \tag{7.1}$$

Second, instead of the closed string boundary conditions $\gamma_i^{(0)}\Big|_{\partial\Sigma} = 0$ and $\gamma^{(0)}\big|_{\partial\Sigma} = 0$, we have the symmetries under $\sigma \to \sigma + 2\pi$ and $\sigma \to -\sigma$, for the open string variables q^i, f.

The relation between the closed and open string variables clarifies the origin of non-commutativity. The closed string variables depend on the open string ones, but also on the corresponding momenta. Hence, the Poisson brackets between dynamical variables become non-trivial on the world-sheet boundary.

Beside the well known coordinate non-commutativity, we established the non-commutativity relation between the Dp-brane coordinates and the conformal part of the world-sheet metric. Both expressions for the non-commutativity parameters, Θ^{ij} and Θ^i, are proportional to the Born–Infeld field strength $\mathcal{F}_{ij}$. In the linear dilaton background, we have $a_i\Theta^{ij} = 0$ and $a_i\Theta^i = 0$, which makes the coordinate corresponding to the a_i direction commutative.

Let us compare the results of the present paper with those of [7]. From the conformal invariance on the boundary, $(T_+ + T_-)|_{\partial\Sigma} = 0$, these authors obtained an additional constraint on the background fields, $a^i\mathcal{F}_{ij} = 0$. In our approach, the boundary condition $(T_+ + T_-)|_{\partial\Sigma} = 0$ is satisfied for arbitrary background fields. In fact, with the help of (5.8), we have $T_\pm = \tilde{T}_\pm$, and the above equation takes the form $\tilde{T}_+ + \tilde{T}_- = 0$. As a consequence of the second relation of (7.1), this condition is satisfied without any restriction on the background fields.

The constraint $a^i\mathcal{F}_{ij} = 0$ of [7] is, in fact, a consequence of the gauge fixing used. In the gauge $F = 0$, the Poisson bracket $\{x^i, F\}$ vanishes, which, according to (4.15), produces the above constraint.

In our treatment, the effective metric tensor $\tilde{G}_{ij}$ and the non-commutativity parameters Θ^{ij} and Θ^i explicitly depend on the dilaton field. Therefore, we have a commutative coordinate in an arbitrary background. In [7], the existence of the commutative direction is a consequence of the condition $a^i\mathcal{F}_{ij} = 0$. In particular, this condition reduces our $\tilde{G}_{ij}$ and Θ^{ij} to those of [7], while Θ^i vanishes. Thus, our results are more general, as they are valid without the above restriction on the background fields.

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